4th Lecture

Measurement of Stress on the Surface of a Body

Thin-walled Pipe Loaded by Torsion and Internal Pressure

Contents:

4.1 – Introduction

In the previous lectures we have shown a possibility of uniaxial stress measurement in long thin bars stressed by tension, pressure and bending. In this lecture we are going to show basic definitions necessary for measuring plane state of stress on the surface of a thin-walled pipe stressed by torsion and internal overpressure. It is suitable to stress that many of the further given definitions and relations are generally valid and do not refer to the thin-walled pipes only.

4.2 – Torsion of a Thin-walled Cylindrical Pipe

In the Fig. 4.1 there is a thin-walled pipe stressed by a torsion moment M_k (N.m).This torsion moment is induced by a couple of forces *F*. It is experimentally proven that the cross-sections perpendicular to the longitudinal axis of the pipe remain perpendicular even after the pipe loading. The cross-sections only angle mutually under the influence of the pipe deformation. The internal forces in the cross-section result in the occurrence of shear stresses τ. For the thin-walled pipe the following equation of equilibrium of internal and external torsion moments can be written:

$$2 \cdot \pi \cdot h \cdot r_s \tau \cdot r_s = M_K \quad (4.1a)$$

$r_s = r + \frac{h}{2}$, where r_s is an average radius.

Herefrom we shall get a relation between an external torque M_K and shear stress:

$$\tau = \frac{M_K}{2\pi \cdot h \cdot r_s^{\,2}} \quad (4.1b)$$

The torque induces the pipe torsion whereas the surface straight line changes into a spiral due to the deformation. From the Fig. 4.1 a relation between the angle φ of the cross-section and the angle γ of the surface line is obvious, i.e.

$$\gamma \cdot l = \varphi \cdot r$$

Hence (4.2)

$$\gamma = \varphi \cdot \frac{r}{l}$$

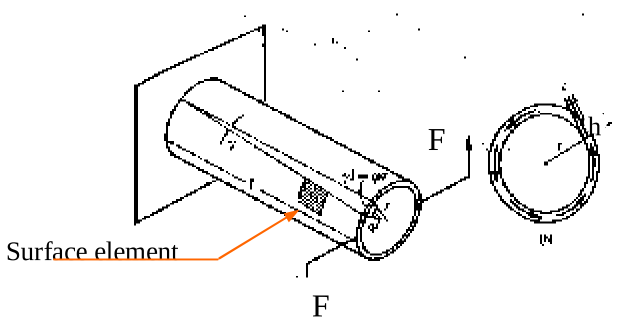

Fig. 4.1

Angle of the surface line due to the torque loading of the pipe is called *shear strain*. The relation between the shear stress and the *shear strain* for a linear elastic body is given by the Hooke's Law for the shear, i.e.

$$t = G \cdot g \tag{4.3}$$

The quantity *G* in the previous relation is the *shear modulus* and is expressed in MPa. There were two material constants introduced for the linear elastic body in the 1st Lecture, namely Young's modulus of elasticity *E* and Poisson's coefficient υ. It can be shown that the shear modulus can be expressed by means of the modulus of elasticity *E* and the Poisson's coefficient υ, by the following relation:

$$G = \frac{E}{2(1+u)} \tag{4.4}$$

Let us withdraw an element from the pipe, crosshatched in the Fig.4.1. The momentum condition of equilibrium of this element results in the so called the *rule of the conjugate shear stresses* showing that the shear stresses on two mutually perpendicular surfaces are of the same size but opposite sign (Fig. 4.2).

$$t_a = t_t = t$$

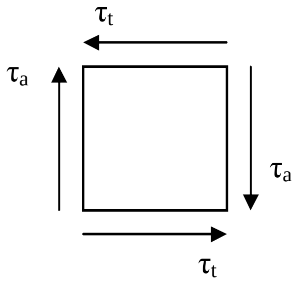

Fig. 4.2

4.3 – Thin-walled Cylindrical Pressure Vessel

In the Fig.4.3 there is schematic picture of a thin-wall pressure vessel stressed by internal overpressure *p*. When analysing the state of stress, we shall limit ourselves to the surface locations sufficiently distant from the vessel fronts.

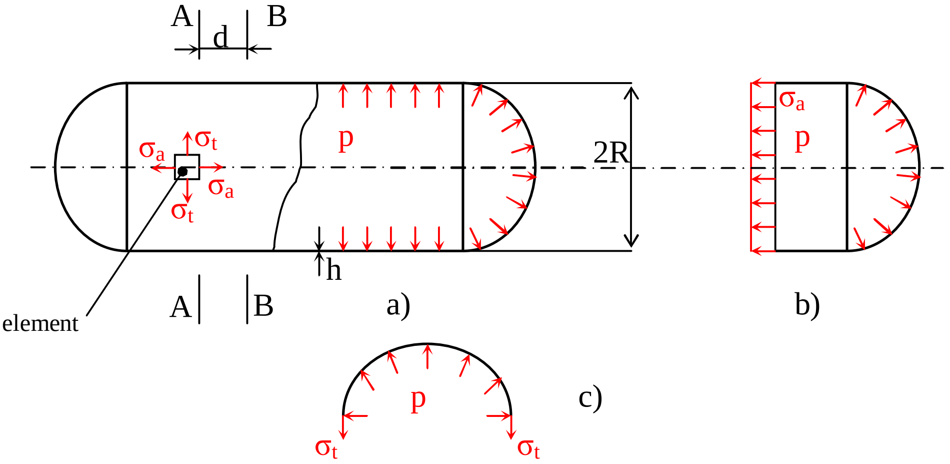

Fig. 4.3

The balance equation for the separated parts of the vessel (Fig. 4.3b) is as follows:

$$s_a \cdot 2p \cdot R \cdot h - p \cdot pR^2 = 0$$

Herefrom

$$s_a = \frac{p \cdot R}{2h} \qquad (4.5)$$

The balance equation of a split ring of the width *d* (Fig. 4.3c) is

$$2s_t \cdot h \cdot d = p \cdot 2R \cdot d$$

Herefrom

$$s_t = \frac{p \cdot R}{h} \qquad (4.6)$$

4.4 – Biaxial Stress – Mohr's Circle

An example of biaxial stress is the state of stress of a thin-walled pipe stressed by internal overpressure or torsion. This state of stress occurs also in other machinery parts, such as plates and shafts. The biaxial state of stress is a special case of plane state of stress. The plane state of stress will be dealt with in detail in the 5th Lecture.

On the element of the thin-walled vessel we have denominated the normal stress in the direction of the axis by the index *a* (axial direction). For the circumferential direction we have used the index *t* (tangential direction). To be more general, let us denominate *a* as the direction *x*. Similarly let us denominate direction *t* as the direction *y*. The element withdrawn from the vessel surface with the new denomination of directions of the normal stresses is given in the Fig. 4.4a. Let us remark that the index shows the direction of the normal to the cross-section plane *pq*, the other index at the shear stress shows the direction of its action.

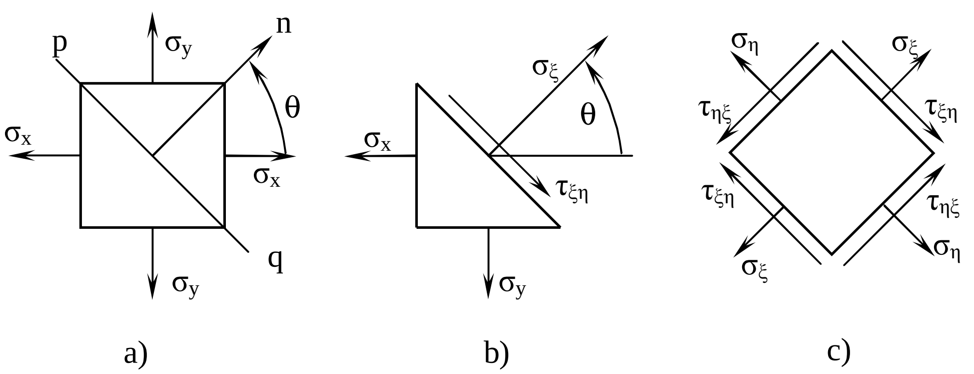

Fig. 4.4

Now let us determine the normal stress σ_ξ and the tangential stress $\tau_{\xi\eta}$ on the cross-section surface *pq* (Fig. 4.4b). From the balance equations of a triangular element (see Fig.4.4b) after conversion we shall get the following relations:

$$s_x = \frac{1}{2}(s_x + s_y) + \frac{1}{2}(s_x - s_y)\cos 2q \ , \qquad (4.7)$$

$$\tau_{\xi\eta} = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \quad . \tag{4.8}$$

If we replace the angle θ in the relations (4.2) by θ + π/2, we shall get the stresses σ_η , $\tau_{\eta\xi}$ acting on the surfaces with the normal η (Fig.4.4c). It is valid:

$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta \quad . \tag{4.9}$$

$$\tau_{\eta\xi} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \quad . \tag{4.10}$$

Addition of the equations (4.7) and (4.9) results in the relation

$$\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y \ , \tag{4.11}$$

according to which the addition of normal stresses at two mutually perpendicular surfaces is constant. From the equations (4.8) and (4.10) we shall get the relation

$$\tau_{\xi\eta} = -\tau_{\eta\xi} \quad , \tag{4.12}$$

which shows that the shear stresses at two mutually perpendicular surfaces are identical but of the opposite sign (the rule of the conjugate shear stresses).

The shear stresses $\tau_{\xi\eta}$ are zero, if θ = 0 and maximal, if θ = π/4. In that case

$$\tau_{max} = \frac{1}{2}(\sigma_x - \sigma_y) \quad . \tag{4.13}$$

The stresses σ_x , σ_y are so called main normal stresses, usually denominated as σ_1 , σ_2. As it is obvious from the Fig. 4.4, there are no shear stresses on the surfaces where the stresses σ_x , σ_y act. If jσ_x and σ_y are identical, there will be no shear stress on any surface of the element.

<u>Mohr's circle for biaxial state of stress</u>

In the relation 4.7 to 4.10 let us denominate

$$\sigma_s = \frac{1}{2}(\sigma_x + \sigma_y) \tag{4.14}$$

Considering the relation (4.13), then the relations (4.7) and (4.8) can be re-written in the form

$$\sigma_\xi - \sigma_s = \tau_{max} \cdot \cos 2\theta \tag{4.15a}$$

$$\tau_{\xi\eta} = -\tau_{max} \cdot \sin 2\theta \tag{4.15b}$$

If the parameter 2θ is eliminated from the previous relations, we shall get the equation of the circle with independent variables σ_ξ and $\tau_{\xi\eta}$. This circle is illustrated in the Fig. 4.5.

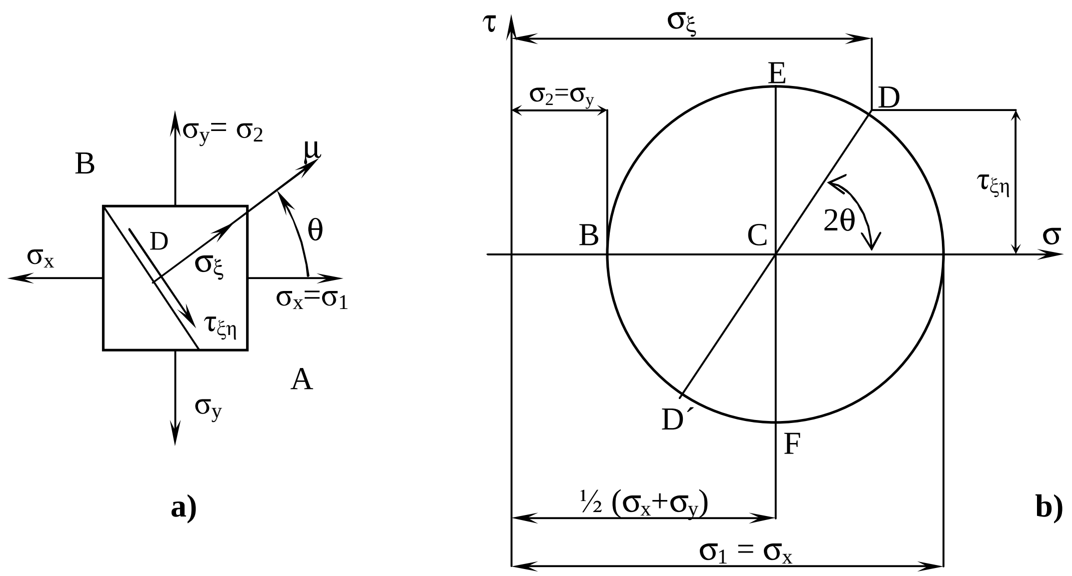

Fig. 4.5

4.5 – Deformation at Biaxial State of Stress –Hooke's Law

Let us return to the state of stress of a thin-walled vessel loaded by internal overpressure *p*. Deformation in the axial direction ε_a evidently depends on both components of the stresses σ_a, σ_t. If each of these stresses acts separately, then axial strain ε_a will be given by superposition of their effects. Similarly this statement is valid for circumferential (tangential) deformation ε_t. i.e.

$$e_a = \frac{1}{E}\left[s_a - u \cdot s_t\right] , \qquad (4.16a)$$

$$e_t = \frac{1}{E}\left[s_t - u \cdot s_a\right] . \qquad (4.16b)$$

Relative change of the vessel wall thickness is then given by the relation

$$e_r = -\frac{u}{E}\left[s_a + s_t\right] . \qquad (4.16c)$$

The stresses σ_a and σ_t can be expressed from the equations (4.16a) and (4.16b) by means of the deformations ε_a, ε_t. i.e.

$$s_a = \frac{E}{1-u^2}\left(e_a + u \cdot e_t\right) , \qquad (4.17a)$$

$$s_t = \frac{E}{1-u^2}\left(e_t + u \cdot e_a\right) . \qquad (4.17b)$$