**<u>1st Lecture</u>**

# Rheological Models of Technical Materials at Simple Tension and Pressure

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## *1.1 – Basic Notions*

### Internal Forces – Stress

A general body in the state of static balance is loaded with external forces. These forces cause the body deformation. This deformation results in so called **internal forces** in the body. Magnitude of the internal forces is determined by the **method of a virtual section**. The bodies with the longitudinal dimension much longer than two other dimensions are called bars. Join of the centres of gravity of the individual sections is called a **bar axis**. If the bar axis is a straight line, the bar is called the **straight** bar. The straight bar is the body of our interest in the 1st lecture.

Direct (normal) stress σ is a normal internal force N related to the size of a cross section $A_0$.

$$s = \frac{N}{A_0} \quad . \tag{1.1}$$

A bar element of the original length dx influenced by the internal forces changes its length to the magnitude dx + Δ dx . The change of the element length dx is thus Δdx. (see fig. 1.1)

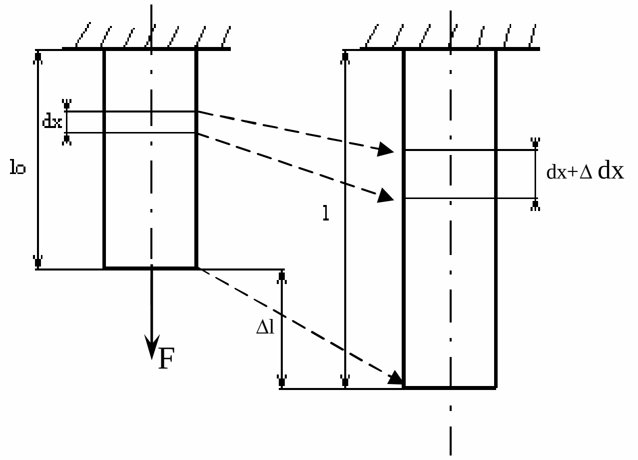


**Fig. 1.1**

**Relative deformation (strain)** is determined by the ration of the length change and the original length, i.e.

$$e\,(x) = \frac{\Delta\, dx}{dx} \quad . \tag{1.2}$$

If along the whole bar length ***e(x) = const***, the relative strain can be expressed from the definitive dimensions of the whole bar, i.e.

$$e = \frac{\Delta l}{l_0} \quad . \tag{1.3}$$

**Normal deformation** is linked with the normal stress. Normal deformation linked with **length extension** is signed as **positive (+)**. The corresponding normal stress is **tensile** and is signed (+). Similarly, the sign (-) is introduced for pressure deformation and **pressure normal stress**.

Axial normal strain of the bar is always linked with the lateral deformation of the opposite sign in two directions perpendicular to the axial deformation.

Absolute value of the **ratio of lateral deformation** and **longitudinal deformation** is called **Poisson's ratio**. This ratio is designated by a small Greek letter $\mu$ or $\nu$.

## *1.2 – Essential Rheological Models of Technical Materials*

**Rheology** is part of mechanics of continuum. In this branch we investigate **general laws of formation and growth of deformation in materials** owing to various causes and in various thermo dynamical conditions. Technically speaking, we look for a **relation between internal forces (stress) and material deformation of the deformed body**.

In case of material of the bar loaded by uniaxial tension and pressure, the basic **rheological models** are the following ones. Let us consider a bar with a unit area cross section.

**a) Linear-elastic model of material**

The model consists of a linear spring

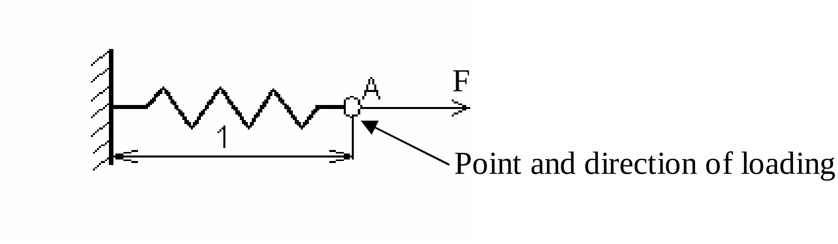


**Fig. 1.2**

Work diagram

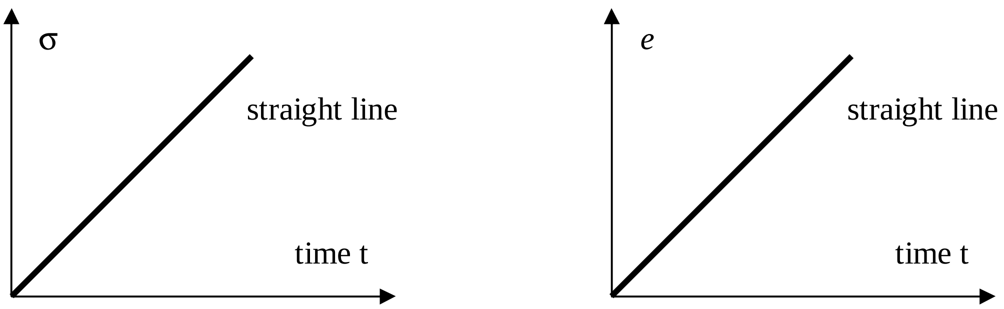


**Fig. 1.3**

When the time is eliminated, we will get dependence

$$\sigma = f\,(e) = E\,e\,. \tag{1.4}$$

This dependence is called **Hooke's law**. The constant of proportionality E is denominated **Young's modulus of elasticity**.

The model of linear elastic material is the most often used model of technical materials, such as steel, *Al* and *Mg* alloys and some short-time loaded plastic materials.

The linear elastic model of materials is suitable also for short-time loaded parts at elevated temperatures. However, it is necessary to consider Young's modulus dependence on temperature.

## b) Ideal viscose model of material

The model is formed by a damper (cataract) with Newton liquid.

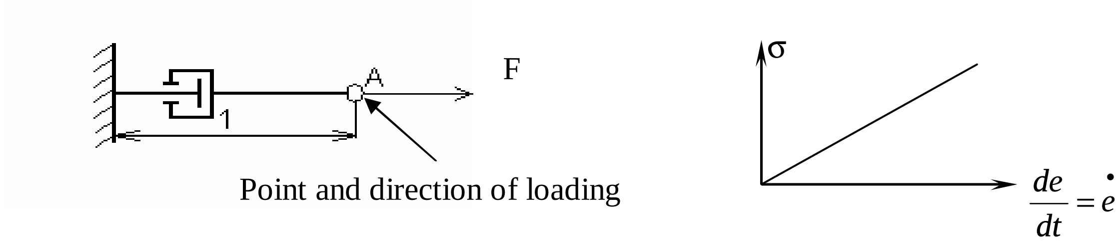


**Fig. 1.4**

Resistance of piston in the cataract grows linearly with the growth of piston displacement speed, i.e.

$$s = h \frac{de}{dt} = h \cdot \dot{e} \ , \qquad (1.5)$$

where $h$ is coefficient of viscosity.

## c) Ideal plastic model of material

The model allows linear motion of a body towed across a plane surface on condition of validity of Coulomb's **friction**.

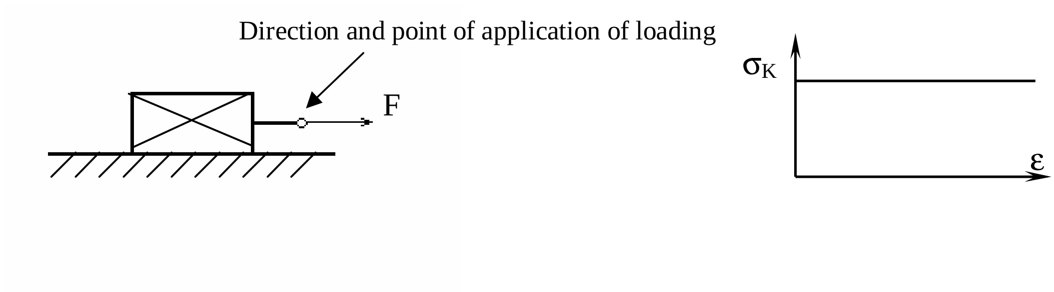


**Fig. 1.5**

Internal force (stress) necessary for overcoming the resistance against the motion is called the **yield point** and in literature is often designated as $\boldsymbol{\sigma_k}$ or $\boldsymbol{\sigma_Y}$.

## *1.3 – Elementary Rheological Models of Creep*

**Creep** is **permanent (non-reversible) deformation** realized at **constant temperature** and **constant stress** (or loading) **depending on time**. Graphically presented time dependence of deformation is called a **creep curve** (see Fig. 1.6).

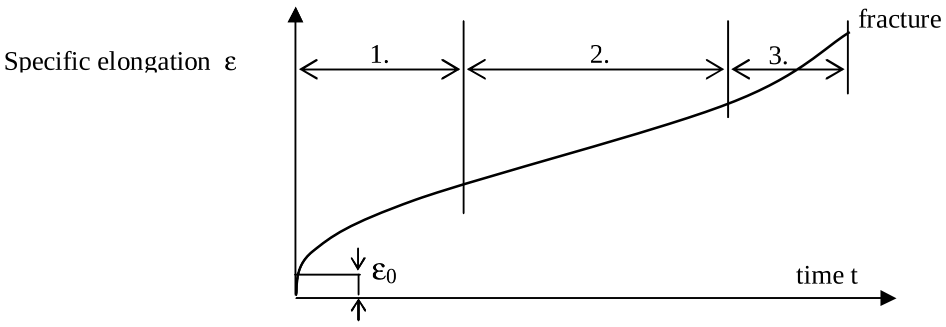


**Fig. 1.6**

This curve describes (vs. time and stress) one, two or three stages of creep. The first stage, where the creep speed goes down following the immediate initial elongation, is called the stage of primary or transient creep. In the second stage the creep speed does not change with time. It is called the stage of secondary creep with the speed εs. In the third stage – the stage of tertiary creep – the creep speed dramatically grows with time. The tertiary stage results in fracture.

Further let us show in what way the creep curve can be modelled by means of elementary rheological models in the primary and secondary stages for a uniaxially stressed bar.

### Voigt – Kelvin's Model

This model is created by parallel connection of a cataract and spring (see Fig. 1.7).

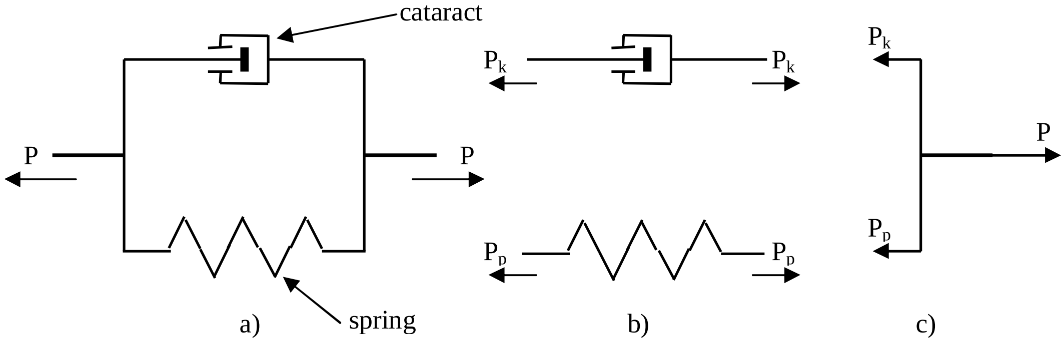


**Fig. 1.7**

Let us designate $\delta_p$ the elongation of the spring. This elongation is proportional to the force $P_p$ acting in the spring, i.e.

$$d_p = k_1 \cdot P_p .$$

Similarly it is true for the cataract: $\frac{dd_k}{dt} = k_2 \,.\, P_k \,.$

It is clear from the Fig. 1.7 that for displacements $\delta_p$ and $\delta_k$ we can write:

$$\delta = \delta_k = \delta_{p..}$$

From the above mentioned figure also validity of the balance equation is obvious:

$$P = P_p + P_k \,.$$

If we replace forces $P_p$ and $P_k$ by the previous relations, we shall get:

$$P = \frac{d}{k_1} + \frac{1}{k_2} \cdot \frac{dd}{dt} \quad . \qquad (1.6)$$

If we substitute the stress $\sigma$ for the forces $P$ with, the deformation $\varepsilon$ for the displacement $\delta$ with and coefficients $k_1 = 1/E$ and $k_2 = 1/\eta$ , then we can write:

$$s = E \cdot e + h \frac{de}{dt} \quad . \qquad (1.7)$$

Considering that the initial deformation is zero, the equation (1.7) can be integrated at constant stress $\sigma$. Thus we shall get the following exponential relation for the deformation $e$.

$$e = \frac{s}{E}\left[1 - \exp\left(-\frac{E}{h} \cdot t\right)\right] . \qquad (1.8)$$

It is clear from the equation (1.8) that for $t \to \infty$ deformation $e$ tends to the value $\sigma/E$. Time behaviour of the deformation for the Voigh-Kelvin’s model of a viscoelastic body is schematically presented in the Fig. 1.8.

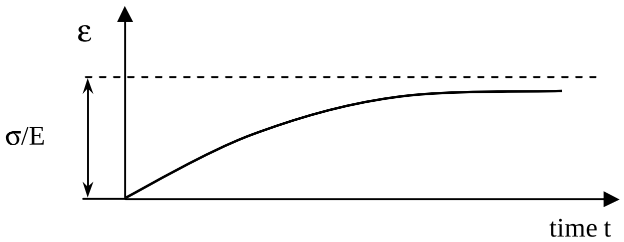


**Fig. 1.8**

## Standard Model

This model consists of two „springs‘‘ and one cataract (damper). Connection of these members is obvious from the Fig. 1.9.

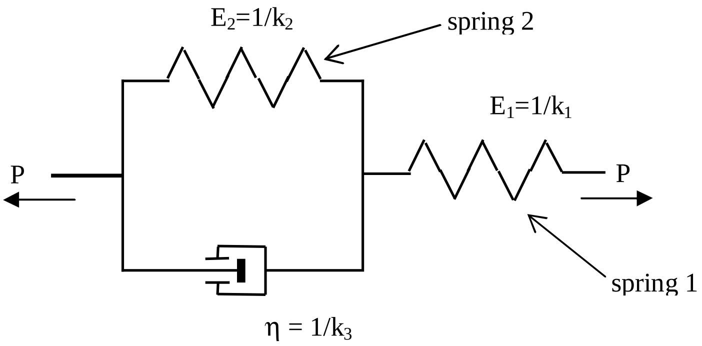


Fig. 1.9

It is valid:

$$\frac{ds}{dt} + a \cdot s = E_1 \left( \frac{de}{dt} + b \cdot e \right) , \tag{1.9}$$

where

$$a = \frac{E_1 + E_2}{h} \quad \text{and} \quad b = \frac{E_2}{h} .$$

At a sudden application of loading, the equation (1.9) comes down to the relation:

$$\frac{ds}{dt} = E_1 \cdot \frac{de}{dt} .$$

Following the integration it is valid: $s = E_1 \cdot e$ .

For t = 0 it is the case of elastic deformation of the spring 1. The modulus $E_1$ is called the immediate modulus of elasticity.

In case of a very slow application of loading the derivation of stress and strain with respect of time is very small in comparison with other members in the equation (1.9). Having neglected the members with time derivations in the equation (1.9) we shall get:

$$s = \frac{E_1 \cdot b}{a} \cdot e = \frac{E_1 \cdot E_2}{E_1 + E_2} \cdot e .$$

The quantity $\dfrac{E_1 \cdot E_2}{E_1 + E_2}$ is often designated as the long-term modulus of elasticity.

It is obvious that this long-term modulus is given by a linear connection of the spring 1 with the spring 2 (see fig.1.9)

Integrating the equation (1.9) for σ = const., we shall get:

$$e = \frac{s}{E_1}\left\{1 + \frac{a-b}{b}\left[1 - \exp(-b \cdot t)\right]\right\} \quad . \tag{1.10}$$

The equations (1.7) and (1.9) are designated as constitutive equations of the given model of the viscoelasic body. It is possible to construct models consisting of many elementary "springs" and "cataracts". The constitution equation of such a viscoelastic body can then be generalized into the following formula:

$$a_0 \cdot s + a_1 \frac{ds}{dt} + \ldots + a_n \cdot \frac{d^n s}{dt^n} = b_0 \cdot e + b_1 \frac{de}{dt} + \ldots + b_n \frac{d^n e}{dt^n} \quad . \tag{1.11}$$

## *1.4 – Elementary Models of Plasticity*

Plasticity is the ability of the body to be deformed permanently non-reversibly and independently of time due to the application of loading.

The rheological model of ideal rigid-plastic material was presented in the chapter 1.2.

Let us remind you that this model consists of a body towed across a horizontal rough surface. The permanent, non-reversible plastic deformation in the rigid-plastic body occurs when exceeding the yield point $\sigma_k$ .

Adding a linear spring in the series (see Fig. 1.10a) to the towed body, a rheological model of an **ideal elasto-plastic material** with the yield strength $\sigma_k$ and modulus of elasticity $E$ is created. Dependence of the stress $\sigma$ on the deformation $\varepsilon$, i.e. deformation characteristic, is given in the Fig. (1.10b).

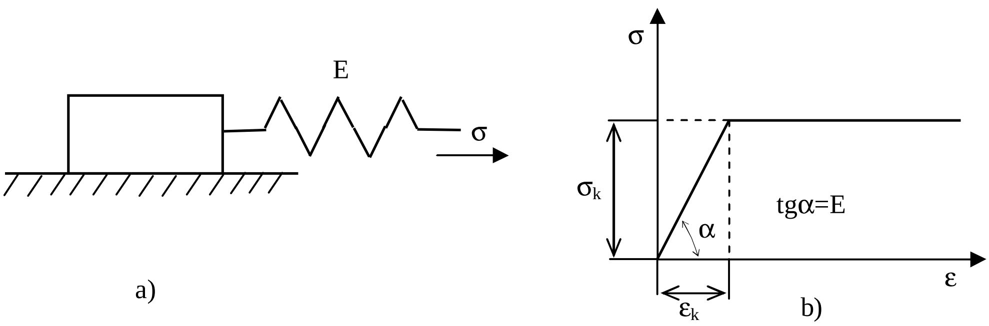


**Fig. 1.10**

Rheological model of an **elastoplastic body with hardening** is given in the Fig..(1.11a).

Deformation characteristic of this model is in the Fig.(1.11b)

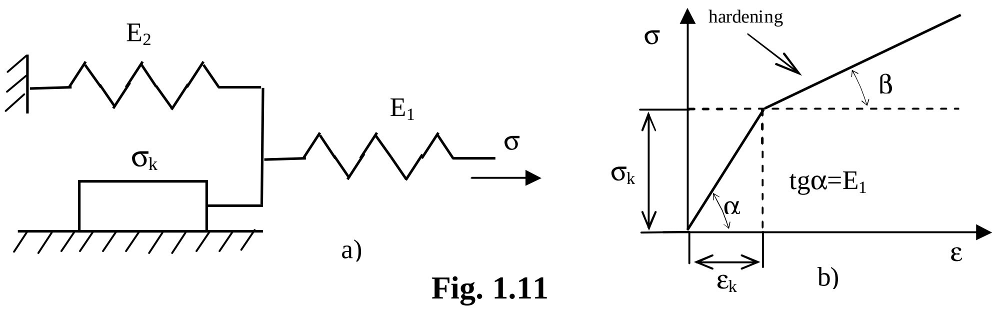


**Fig. 1.11**

In the Fig. 1.11 means:

$$tga = E_1 \qquad for\ \ s \le s_k\ ,$$

$$tgb = \frac{E_1 \cdot E_2}{E_1 + E_2} \qquad for\ \ s \ge s_k\ .$$

## *1.5 – Rheological Model of Elasto-Visco-Plastic Material*

During the deformation wave propagation or at high temperature low-cycle fatigue the viscous and plastic deformations can be mutually bound (viscoplastic material). Providing elastic response to the loading occurs too, an elastic-viscoplastic model can be created by means of the basic rheological models (see chapt.1.2) which is presented in the Fig. 1.12.

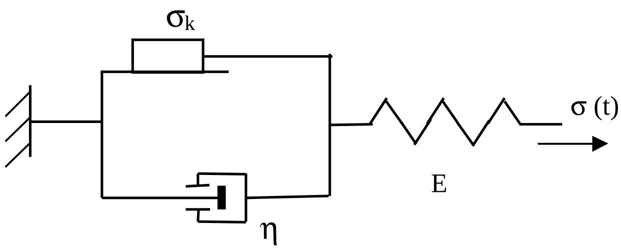


**Fig. 1.12**

Constitutive equations of this model have to distinguish the cases when the plastic deformations occur and when not.

It is valid:

$$\frac{de}{dt} = \frac{1}{E} \cdot \frac{ds}{dt} + \frac{1}{h}\left(s \pm s_k\right) \quad pro\left|s\right| \ge s_k\ \ , \qquad (1.12a)$$

$$\frac{de}{dt} = \frac{1}{E} \cdot \frac{ds}{dt} \qquad pro\left|s\right| < s_k\ \ . \qquad (1.12b)$$

In the equation 1.12a the upper sign is valid for tension (σ>0) and lower sign is valid for pressure (σ<0).